

Nombres complexes : activité et exercices

Ex 1

Mettre sous forme algébrique les expressions suivantes sachant que $z = 2 + 3i$ et $z' = -1 + i$

$$\begin{array}{ll}
 \text{a)} z + z' = 2 + 3i - 1 + i = 1 + 4i & \text{b)} 2z - 3z' = 2(2 + 3i) - 3(-1 + i) = 4 + 6i + 3 - 3i = 7 + 3i \\
 \text{c)} z \times z' = (2 + 3i) \times (-1 + i) = -2 + 2i - 3i - 3 = -5 - i & \text{d)} z'^2 = (-1 + i)^2 = 1 - 2i - 1 = -2i \\
 \text{e)} z^3 = (2 + 3i)^2 (2 + 3i) = (4 + 12i - 9)(2 + 3i) = (-5 + 12i)(2 + 3i) = -10 - 15i + 24i - 36 = -46 + 9i \\
 \text{f)} (1 + z)(1 + z') = (3 + 3i)(i) = 3i - 3 & \text{g)} \frac{1}{z} = \frac{1}{2+3i} = \frac{(2-3i)}{(2+3i)(2-3i)} = \frac{(2-3i)}{4+9} = \frac{2-3i}{13} \\
 \text{h)} \frac{1}{z'} = \frac{1}{-1+i} = \frac{(-1-i)}{(-1+i)(-1-i)} = \frac{-1-i}{2} & \text{i)} z/z' = z \times (1/z') = (2 + 3i) \frac{-1-i}{2} = \frac{-2-2i-3i+3}{2} = \frac{1-5i}{2} \\
 \text{j)} 1/z^2 = (1/z)^2 = \left(\frac{2-3i}{13}\right)^2 = \frac{4-12i-9}{169} = \frac{-5-12i}{169} & \\
 \text{k)} (1 + z')/(1 - z') = \frac{i}{(2-i)} = \frac{i(2+i)}{(2-i)(2+i)} = \frac{2i-1}{5} = \frac{3+9i}{5} &
 \end{array}$$

Ex 2

Déterminer z_0 , z_1 , et z_2 éléments de \mathbb{C} tels que :

$$\begin{array}{lll}
 \text{a)} \frac{z_0+2}{z_0-3} = 2 - 3i & \text{b)} \begin{cases} 2z_1 + z_2 = 4 \\ -2iz_1 + z_2 = 0 \end{cases} & \Leftrightarrow \begin{cases} 2z_1 + z_2 = 4 \\ -2(i+1)z_1 = -4 \end{cases} \\
 z_0 + 2 = (2-3i)(z_0 - 3) & \Leftrightarrow \begin{cases} 2z_1 + z_2 = 4 \\ z_1 = \frac{-4}{-2(i+1)} \end{cases} & \Leftrightarrow \begin{cases} 2z_1 + z_2 = 4 \\ z_1 = \frac{2(i-1)}{(i+1)(i-1)} \end{cases} \\
 z_0 + 2 = 2z_0 - 6 - 3iz_0 + 9i & \Leftrightarrow \begin{cases} 2z_1 + z_2 = 4 \\ z_1 = \frac{2(i-1)}{-2} \end{cases} & \Leftrightarrow \begin{cases} 2z_1 + z_2 = 4 \\ z_1 = 1 - i \end{cases} \\
 z_0 - 2z_0 - 3iz_0 = -8 + 9i & \Leftrightarrow \begin{cases} z_2 = 4 - 2(1 - i) \\ z_1 = 1 - i \end{cases} & \Leftrightarrow \begin{cases} z_2 = 2 + 2i \\ z_1 = 1 - i \end{cases} \\
 z_0(-1-3i) = -8 + 9i & & \\
 z_0 = \frac{-8+9i}{(-1-3i)} = \frac{(-8+9i)(-1+3i)}{(-1-3i)(-1+3i)} = \frac{8-24i-9i-27}{1+9} = \frac{-19-33i}{10} & &
 \end{array}$$

Ex 3

Déterminer les modules et arguments des nombres complexes suivants

$$\begin{array}{lllll}
 \text{a)} 1 - i & \text{b)} 2 + 2i & \text{c)} -1 + i\sqrt{3} & \text{d)} -2i & \text{e)} -3 \\
 \rho = \sqrt{1^2 + (-1)^2} & \rho = \sqrt{2^2 + 2^2} & \rho = \sqrt{(-1)^2 + \sqrt{3}^2} & \rho = \sqrt{(-2)^2} & \rho = \sqrt{3^2} \\
 \rho = \sqrt{2} & \rho = 2\sqrt{2} & \rho = \sqrt{4} = 2 & \rho = 2 & \rho = 3 \\
 z = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} & z = 2\sqrt{2} \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) & z = 2 \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) & z = 2(0 - i) & z = 3(-1 + 0i) \\
 \begin{cases} \cos\theta = \frac{\sqrt{2}}{2} \\ \sin\theta = -\frac{\sqrt{2}}{2} \end{cases} & \begin{cases} \cos\theta = \frac{\sqrt{2}}{2} \\ \sin\theta = \frac{\sqrt{2}}{2} \end{cases} & \begin{cases} \cos\theta = \frac{-1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2} \end{cases} & \begin{cases} \cos\theta = 0 \\ \sin\theta = -1 \end{cases} & \begin{cases} \cos\theta = -1 \\ \sin\theta = 0 \end{cases} \\
 \begin{cases} \cos\theta = \cos\frac{\pi}{4} \\ \sin\theta = -\sin\frac{\pi}{4} \end{cases} & \begin{cases} \cos\theta = \cos\frac{\pi}{4} \\ \sin\theta = \sin\frac{\pi}{4} \end{cases} & \begin{cases} \cos\theta = -\cos\frac{\pi}{3} \\ \sin\theta = \sin\frac{\pi}{3} \end{cases} & \begin{cases} \cos\theta = \cos\frac{\pi}{2} \\ \sin\theta = -\sin\frac{\pi}{2} \end{cases} & \begin{cases} \cos\theta = -\cos 0 \\ \sin\theta = \sin 0 \end{cases} \\
 \theta = -\frac{\pi}{4} & \theta = \frac{\pi}{4} & \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3} & \theta = -\frac{\pi}{2} & \theta = \pi
 \end{array}$$

$$\begin{array}{l}
 \text{f)} -\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right) \\
 \rho = \sqrt{\left(-\cos\left(\frac{\pi}{6}\right)\right)^2 + \sin^2\left(\frac{\pi}{6}\right)} = 1 \\
 \begin{cases} \cos\theta = -\cos\left(\frac{\pi}{6}\right) \\ \sin\theta = -\sin\left(\frac{\pi}{6}\right) \end{cases} \\
 \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}
 \end{array}$$

$$\begin{array}{ll}
 \text{g)} 3 + i & \text{h)} 4 - 2i \\
 \rho = \sqrt{10} \approx 3,16 & \rho = \sqrt{20} \approx 4,47 \\
 \theta = \cos^{-1}\left(\frac{3}{\sqrt{10}}\right) & \theta = -\cos^{-1}\left(\frac{4}{\sqrt{20}}\right) \\
 \approx 0,32 \text{ rad} & \approx -0,46 \text{ rad}
 \end{array}$$

Ex 4

Déterminer les parties réelles et imaginaires des complexes tels que :

$$\begin{array}{llll}
 \text{a) } \rho = 3 \text{ et } \theta = \frac{\pi}{3} & \text{b) } \rho = 2 \text{ et } \theta = -\frac{\pi}{6} & \text{c) } \rho = 1 \text{ et } \theta = \frac{7\pi}{3} & \text{d) } \rho = 4 \text{ et } \theta = \frac{3\pi}{4} \\
 z = 3(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)) & z = 2(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)) & z = \cos\left(\frac{7\pi}{3}\right) + i\sin\left(\frac{7\pi}{3}\right) & z = 4(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)) \\
 = 3\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) & = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) & = \left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) & = 4\left(-\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\
 = \frac{3}{2} + i\frac{3\sqrt{3}}{2} & = \sqrt{3} - i & = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) & = -2\sqrt{2} + 2\sqrt{2}i \\
 & & & \text{Car } \frac{3\pi}{4} = \pi - \frac{\pi}{4}
 \end{array}$$

Ex 5

En vous inspirant des résultats de l'exercice 3, déterminer les modules et arguments des complexes suivants :

$$(2+2i)/(-1+i\sqrt{3}); \quad 1/(-2i); \quad (1-i)(2+2i) \quad \left(-\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)^3$$

On utilisera les règles : $\arg(z \times z') = \arg(z) + \arg(z')$ et $|z \times z'| = |z| \times |z'|$
 $\arg(z/z') = \arg(z) - \arg(z')$ et $|z/z'| = |z| / |z'|$

Ainsi pour $(2+2i)/(-1+i\sqrt{3})$ $\rho = \frac{2}{2} = 1$ et $\theta = \arg(2+2i) - \arg(-1+i\sqrt{3}) = \frac{\pi}{4} - \frac{2\pi}{3} = \frac{-5\pi}{12}$
 $1/(-2i)$ $\rho = \frac{1}{2}$ et $\theta = \arg(1) - \arg(-2i) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$
 $(1-i)(2+2i)$ $\rho = \sqrt{2} \times 2\sqrt{2} = 4$ et $\theta = \arg(1-i) - \arg(2+2i) = -\frac{\pi}{4} + \frac{\pi}{4} = 0$
 $\left(-\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)^3$ $\rho = 1^3 = 1$ et $\theta = 3 \times \arg\left(-\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right) = 3 \cdot \frac{7\pi}{6} = \frac{7\pi}{2} = \frac{3\pi}{2}$

Ex 6

Soit A, B et C des points d'affixes respectives $2+\frac{7}{2}i$, $5+\frac{1}{2}i$ et $4+\frac{11}{2}i$

a) déterminer l'affixe de I le milieu de [AB]. b) Faire une figure c) Calculez l'affixe des vecteurs \vec{AB} , \vec{AC} et \vec{CB}

d) En déduire les distances AB, AC et BC e) Que peut-on dire du triangle ABC

a) $z_i = \frac{z_A+z_B}{2} = \frac{2+\frac{7}{2}i+5+\frac{1}{2}i}{2} = \frac{7+4i}{2} = 3,5 + 2i$

c) $z_{\vec{AB}} = \frac{z_B-z_A}{2} = 5 + \frac{1}{2}i - 2 - \frac{7}{2}i = 3 - \frac{6}{2}i = 3 - 3i$

$$z_{\vec{AC}} = \frac{z_C-z_A}{2} = 4 + \frac{11}{2}i - 2 - \frac{7}{2}i = 2 + \frac{4}{2}i = 2 + 2i$$

$$z_{\vec{CB}} = \frac{z_B-z_C}{2} = 5 + \frac{1}{2}i - 4 - \frac{11}{2}i = 1 - \frac{10}{2}i = 1 - 5i$$

d) $AB = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$ $AC = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ $CB = \sqrt{1^2 + 5^2} = \sqrt{26}$

e) On remarque que $CB^2 = AB^2 + AC^2$ donc d'après la réciproque du théorème de Pythagore ABC est rectangle en A

Ex 7

$z_A = \sqrt{3} + i$, $z_B = \sqrt{3} - i$, $z_C = 2\sqrt{3} + 2i$ et $z_D = 2i$

- a) Déterminer le module et l'argument de chacun de ces nombres.
- b) Dans le plan rapporté à un repère orthonormal $(O; \vec{u}, \vec{v})$ d'unité graphique 2cm, placer les points A,B,C et D d'affixes respectives : z_A, z_B, z_C et z_D
- c) Montrer que les points O, B,C et D sont sur un même cercle de centre A dont on précisera le rayon.

$$\begin{array}{llll}
 \text{a) } |z_A| = \sqrt{\sqrt{3}^2 + 1^2} = 2 & |z_B| = \sqrt{\sqrt{3}^2 + 1^2} = 2 & |z_C| = \sqrt{(2\sqrt{3})^2 + 2^2} = 4 & |z_D| = 2 \\
 z_A = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) & z_B = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) & z_C = 4\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) & \text{trivial} \\
 = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) & = 2\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right) & = 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) & \arg(z_D) = \frac{\pi}{2} \\
 \theta = \frac{\pi}{3} & \theta = -\frac{\pi}{3} & \theta = \frac{\pi}{3}
 \end{array}$$

$$OA = |z_{\vec{OA}}| = |z_A| = 2 \quad AB = |z_{\vec{BA}}| = |z_A - z_B| = |\sqrt{3} + i - \sqrt{3} - i| = |2i| = 2$$

$$AC = |z_{\vec{CA}}| = |z_A - z_C| = |\sqrt{3} + i - 2\sqrt{3} - 2i| = |-\sqrt{3} + i| = |z_B| = 2$$

$$AD = |z_{\vec{DA}}| = |z_A - z_D| = |\sqrt{3} + i - 2i| = |\sqrt{3} - i| = |z_B| = 2$$

Donc AO = AC = AB = AD = 2 donc O,C,B et D sont sur le cercle de centre A et de rayon 2